

Curiosities at $c = -2$ Horst G. Kausch¹*Department of Applied Mathematics and Theoretical Physics,
University of Cambridge, Silver Street,
Cambridge CB3 9EW, U.K.*

20 October 1995

Abstract

Conformal field theory at $c = -2$ provides the simplest example of a theory with “logarithmic” operators. We examine in detail the (ξ, η) ghost system and Coulomb gas construction at $c = -2$ and show that, in contradistinction to minimal models, they can not be described in terms of conformal families of *primary* fields alone but necessarily contain reducible but indecomposable representations of the Virasoro algebra. We then present a construction of “logarithmic” operators in terms of “symplectic” fermions displaying a global $SL(2)$ symmetry. Orbifolds with respect to finite subgroups of $SL(2)$ are reminiscent of the *ADE* classification of $c = 1$ modular invariant partition functions, but are isolated models and not linked by massless flows.

1 Introduction

Two-dimensional conformally invariant field theories feature prominently in two areas of theoretical physics: they provide the perturbative vacua of string theory and they serve as a framework for understanding second order phase transitions of statistical systems. The success of conformal field theory is due to the fact that in two dimensions conformal transformations coincide with analytic coordinate transformations generated by the holomorphic and anti-holomorphic components of the stress tensor. This results in an infinite-dimensional local conformal algebra consisting of two commuting copies of the Virasoro algebra. The conformal anomaly c , called the central charge of the Virasoro algebra, is the main parameter characterising a conformal field theory. In their seminal paper [1] Belavin, Polyakov and Zamolodchikov showed that for the minimal series, $c = 1 - 6(p - p')^2/pp'$

¹Email: H.G.KAUSCH@DAMTP.CAM.AC.UK

with coprime integers $p, p' > 1$, there are only a finite number of irreducible representations of the Virasoro algebra. In such a situation the space of states decomposes into representations of the left and right Virasoro algebra and the partition function can be written as a finite sum over holomorphic times anti-holomorphic characters. A conformal field theory with periodic boundary conditions can naturally be thought of as defined on a torus. This implies the invariance of the partition function under modular transformations. Using this requirement Cappelli, Itzykson and Zuber were able to classify the partition functions for the minimal models [2, 3].

Going beyond the minimal series the simplest example of a conformal field theory is provided by a free massless scalar field. The Virasoro algebra generated from the stress-energy tensor has central charge $c = 1$. If one compactifies the scalar field to take values on a circle of radius ρ the primary fields are classified by their momenta and winding numbers which, by locality, are forced to lie on a lattice resulting in the Coulomb gas partition function $Z(\rho)$. At the radius $\rho = 1$ one obtains the $SU(2)$ WZW model at level one which possesses a global $SU(2)$ symmetry. In this situation one can form orbifold models [4] for any subgroup $\Gamma \subset SU(2)$ introducing additional sectors with twisted boundary. From the A-D-E classification of finite subgroups of $SU(2)$ [5] one can obtain a classification of $c = 1$ modular invariant partition functions [6, 7].

Attempts to study more general conformal field theories beyond the above examples have produced a wealth of examples but their understanding is still rather sketchy. In this paper we shall consider models at $c = -2$, in particular the fermionic (ξ, η) system. It is related to the continuum limit of dense polymers [8] and is generated by two fermionic ghost fields of weight zero and one. It is also of interest for the ghost sector of superstring theory [9]. The system has a $U(1)$ symmetry allowing the introduction of twist fields. The partition function of the (ξ, η) system with \mathbf{Z}_{2N} twist is the same as that of a free boson compactified on a circle of radius $N\sqrt{2}$, however with a different choice of vacuum state. The system is thus modular invariant but has a charge asymmetry and non-vanishing correlation functions require the insertion of a ξ zero-mode. The model also contains fields which are neither primary fields nor descendants of a primary field. They form reducible but indecomposable representations of the Virasoro algebra.

The same features occur in the Coulomb gas construction of minimal models where one introduces screening charges and performs a BRST projection to remove the reducible representations. In the case of the (ξ, η) system the analogous procedure is to go to the kernel of η_0 . This defines a “small” algebra generated by $\partial\xi$ and η . On the “small” algebra the $U(1)$ symmetry is enhanced to an $SL(2)$ symmetry. This symmetry is only global and hence not generated by a Kac-Moody algebra but related to a W-algebra. Twisting the “small” algebra results in characters which do not transform in any simple way under the modular group.

The 4pt function of the twist field calculated from the null vector and monodromy constraints has a logarithmic singularity. This implies that the fusion of twist fields in the “small” algebra results in fields forming two-dimensional Jordan cells for L_0 [10]. These “logarithmic” operators extend the space of fields of the “small” algebra and we present a construction in terms of “symplectic” fermions displaying a $SL(2)$ symmetry. We construct

orbifolds with respect to finite subgroups of $SL(2)$, following the procedure employed in the $c = 1$ case [6], and investigate the effect of perturbing the orbifold models by marginal operators.

2 The (ξ, η) -ghost system

The (ξ, η) system is defined by the action

$$S = \frac{1}{2\pi} \int d^2z \left(\eta \bar{\partial} \xi + \bar{\eta} \partial \bar{\xi} \right), \quad (1)$$

where ξ and η denote holomorphic fermionic ghost fields of dimension 0 and 1. The corresponding operator product expansion is

$$\xi(z)\eta(w) = \eta(z)\xi(w) = \frac{1}{z-w} + O(1). \quad (2)$$

Equivalent relations are satisfied by anti-holomorphic ghost fields $\bar{\xi}$ and $\bar{\eta}$. When considering chiral fields we shall frequently omit mention of the anti-holomorphic components. Introducing the mode expansion

$$\xi(z) = \sum_{n \in \mathbf{Z}} \xi_n z^{-n}, \quad \eta(z) = \sum_{n \in \mathbf{Z}} \eta_n z^{-n-1}, \quad (3)$$

we obtain from the short distance singularity (2) the anti-commutator

$$\{\xi_m, \eta_n\} = \delta_{m+n, 0}, \quad (4)$$

while the other anti-commutators vanish. The stress tensor has central charge $c = -2$ and is given by

$$T(z) = \sum_{n \in \mathbf{Z}} L_n z^{-n-2} = :\partial \xi(z) \eta(z):, \quad (5)$$

where the $:\cdots:$ denotes fermionic normal ordering, that is, the annihilation modes are moved to the right with a minus sign whenever two fields are interchanged. The $sl(2, \mathbf{C})$ -invariant vacuum Ω is characterised by $\xi_m \Omega = 0$, $\eta_n \Omega = 0$, for $m > 0, n \geq 0$.

The (ξ, η) system has a $U(1)$ symmetry which is generated by the natural $U(1)$ current

$$J(z) = :\xi(z) \eta(z):, \quad (6)$$

which counts ξ with charge 1 and η with charge -1 . We can thus consider twist fields σ_λ such that

$$\xi(e^{2\pi i} z) \sigma_\lambda(0) = e^{2\pi i \lambda} \xi(z) \sigma_\lambda(0), \quad (7)$$

$$\eta(e^{2\pi i} z) \sigma_\lambda(0) = e^{-2\pi i \lambda} \eta(z) \sigma_\lambda(0). \quad (8)$$

This implies that acting on the twisted sector \mathcal{M}_λ generated by σ_λ the fermion fields have boundary conditions in the angular direction

$$\xi \mapsto e^{2\pi i \lambda} \xi, \quad \eta \mapsto e^{-2\pi i \lambda} \eta.$$

and a mode expansion

$$\xi(z) = \sum_{m \in \mathbf{Z}} \xi_{m-\lambda} z^{-m+\lambda}, \quad \eta(z) = \sum_{m \in \mathbf{Z}} \eta_{m+\lambda} z^{-m-1-\lambda}. \quad (9)$$

The operator product expansion of $\xi(z)$ and $\eta(w)$ remains the same when acting on a twisted representation. By multiplying it with appropriate powers of z and w ,

$$z^{-\lambda} w^\lambda \xi(z) \eta(w) = z^{-\lambda} w^\lambda \left(\frac{1}{z-w} + J(w) + (z-w)T(w) + \dots \right), \quad (10)$$

we get an equation which can be expanded in integral powers of z and w . Applying the usual contour deformation argument we obtain

$$J(z) = :\xi(z)\eta(z):_\lambda + \lambda z^{-1}, \quad (11)$$

$$T(z) = :\partial\xi(z)\eta(z):_\lambda + \frac{\lambda(\lambda-1)}{2} z^{-2}. \quad (12)$$

In the normal ordered products all annihilation modes $\xi_{m-\lambda}$ and $\eta_{m-1+\lambda}$ with $m > 0$ are moved to the right. We denote the vacuum state with respect to this normal ordering by σ_λ . It has $U(1)$ -charge λ and conformal weight

$$h_\lambda = \frac{\lambda(\lambda-1)}{2}. \quad (13)$$

Shifting the value of λ by one does not change the boundary conditions and results in equivalent expressions for J and T since the shift in the scalar terms is compensated by a change in the normal ordering prescription. Thus, σ_λ and $\sigma_{\lambda+1}$ are in the same twisted sector \mathcal{M}_λ . The twist field σ_λ with $0 \leq \lambda < 1$ is the ground state of the twisted sector. The other fields $\sigma_{\lambda+n}$ are excited twist fields which can be obtained from the ground states by acting with creation modes $\xi_{m-\lambda}$ and $\eta_{m-1+\lambda}$ with $m \leq 0$. When $\lambda \notin \mathbf{Z}$ there are no zero modes of ξ and η and the ground state is non-degenerate, while for $\lambda \in \mathbf{Z}$ we obtain the vacuum representation with ground states $\Omega = \sigma_0$ and $\xi_0 \Omega = \sigma_1$. To count the operator content of the various sectors we introduce the $U(1) \times \text{Vir}$ character

$$\begin{aligned} d_{\mu,\lambda}(\tau) &= \text{tr}_{\mathcal{M}_\lambda} \left(e^{2\pi i \mu J_0} q^{L_0 - \frac{c}{24}} \right) \\ &= e^{-2\pi i \mu \lambda} q^{(1-6\lambda(1-\lambda))/12} \prod_{n=1}^{\infty} \left(1 + e^{2\pi i \mu} q^{n+\lambda-1} \right) \left(1 + e^{-2\pi i \mu} q^{n-\lambda} \right) \\ &= \eta(\tau)^{-1} e^{-2\pi i \mu \lambda} \sum_{m \in \mathbf{Z}} e^{2\pi i \mu m} q^{\frac{1}{2}(m+\lambda-\frac{1}{2})^2}, \end{aligned} \quad (14)$$

where $q = \exp(2\pi i \tau)$ and $\eta(\tau)$ is the Dedekind function,

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n). \quad (15)$$

For the full theory we have to take into account both the left and right movers. We shall exclusively consider diagonal theories so that the contribution to the partition function arising from a sector with boundary conditions μ and λ is

$$D_{\mu,\lambda}(\tau) = |d_{\mu,\lambda}(\tau)|^2. \quad (16)$$

These D functions also arise as determinants of the Laplacian in the calculation of a free field partition function with boundary conditions $-\exp(2\pi i\mu)$ and $-\exp(2\pi i\lambda)$, see e.g. [11].

The partition function of the (ξ, η) system twisted by a cyclic subgroup of $U(1)$ is obtained by summing over all (twisted) sectors and keeping only the twist-invariant states. If we twist by a cyclic group \mathcal{C}_{2N} of *even* order, only bosonic states survive in the partition function which can then be obtained by summing the contributions of the left and right movers over all boundary conditions,

$$Z_N(\tau) = \frac{1}{2N} \sum_{k,l=0}^{2N-1} D_{k/(2N), l/(2N)}(\tau) = \frac{1}{|\eta(\tau)|^2} \sum_{n=0}^{4N^2-1} |\Theta_{n, 2N^2}(\tau)|^2. \quad (17)$$

The last equality expresses the partition function as a sum of theta functions (see Appendix A),

$$\Theta_{n,m}(\tau) = \sum_{k \in \mathbf{Z}} q^{m(k + \frac{n}{2m})^2}. \quad (18)$$

Defining as in [11] the Coulomb gas partition function at radius ρ as

$$Z(\rho, \tau) = \frac{1}{|\eta(\tau)|^2} \sum_{m,n \in \mathbf{Z}} q^{\frac{1}{4}(m\rho+n/\rho)^2} \bar{q}^{\frac{1}{4}(m\rho-n/\rho)^2}, \quad (19)$$

we find

$$Z_N(\tau) = Z(N\sqrt{2}, \tau). \quad (20)$$

In the normalisation of [6] this corresponds to a Coulomb gas partition function at radius $r = 2N$. The partition function is modular invariant as can be seen from the transformation formulae for theta functions.

If we twist by a cyclic group \mathcal{C}_N of *odd* order N the partition function

$$Z_N^{NS}(\tau) = \frac{1}{N} \sum_{k,l=0}^{N-1} D_{\frac{k}{N}, \frac{l}{N}}(\tau) \quad (21)$$

obtained by summing over \mathcal{C}_N boundary conditions still contains fermions. It is invariant under the subgroup of the modular group generated by T and ST^2S . The fermions are Neveu-Schwarz fields, single-valued both on the plane and on the cylinder. To obtain a modular invariant partition function we have to consider the possible spin structures,

$$\tilde{Z}_N^{NS}(\tau) = \frac{1}{N} \sum_{k,l=0}^{N-1} D_{\frac{k}{N} + \frac{1}{2}, \frac{l}{N}}(\tau), \quad (22)$$

$$Z_N^R(\tau) = \frac{1}{N} \sum_{k,l=0}^{N-1} D_{\frac{k}{N}, \frac{l}{N} + \frac{1}{2}}(\tau), \quad (23)$$

$$\tilde{Z}_N^R(\tau) = \frac{1}{N} \sum_{k,l=0}^{N-1} D_{\frac{k}{N} + \frac{1}{2}, \frac{l}{N} + \frac{1}{2}}(\tau). \quad (24)$$

In the Ramond sector the fermions are double-valued on the plane and the cylinder. The partition functions \tilde{Z}_N^{NS} and \tilde{Z}_N^R have an additional insertion of the chirality operator $(-1)^F$. Summing over all spin structures results in the partition function of the bosonic \mathcal{C}_{2N} model,

$$Z_N(\tau) = Z_N^{NS}(\tau) + \tilde{Z}_N^{NS}(\tau) + Z_N^R(\tau) + \tilde{Z}_N^R(\tau), \quad (25)$$

expressing the fact that twisting by $\mathbf{Z}_2 = \{1, (-1)^F\}$ results in $\mathcal{C}_{2N} = \mathbf{Z}_2 \times \mathcal{C}_N$ for odd N .

2.1 Bosonisation

The connection between the (ξ, η) -system and a Coulomb gas goes beyond an equality of partition functions. It extends to correlation functions and operators as well. To go to the Coulomb gas formalism we bosonise the chiral (ξ, η) system using the natural $U(1)$ -current (6),

$$J(z) = i\partial x(z), \quad (26)$$

where $x(z)$ is (the left moving component of) a free scalar field with propagator

$$\langle x(z)x(w) \rangle = -\ln(z-w). \quad (27)$$

Because of the logarithmic propagator $x(z)$ is not itself a Virasoro primary field but derivatives and Wick ordered exponentials of $x(z)$ are. The holomorphic ghost fields can be realised as

$$\xi(z) = :e^{ix(z)}:, \quad \eta(z) = :e^{-ix(z)}:, \quad (28)$$

such that the stress tensor (5) takes the familiar Feigin-Fuchs form,

$$T(z) = -\frac{1}{2}(\partial x(z))^2 + \frac{i}{2}\partial^2 x(z). \quad (29)$$

The twist fields σ_λ are represented by the vertex operators $:\exp(i\lambda x(z)):$ with conformal weight $\lambda(\lambda-1)/2$. The twisted sector \mathcal{M}_λ is given as the sum of Fock spaces \mathcal{F}_μ with $\mu \in \lambda + \mathbf{Z}$. This reproduces exactly the characters, correlation functions and operator formalism of the (ξ, η) system. We proceed analogously to define anti-chiral fields $\bar{x}, \bar{\xi}, \bar{\eta}$.

In the Coulomb gas picture one considers the free scalar field $X(z, \bar{z}) = (x(z) + \bar{x}(\bar{z}))/\sqrt{2}$ with propagator $\langle X(z, \bar{z})X(w, \bar{w}) \rangle = -\ln|z-w|$. It is assumed to be an angular variable, $X \equiv X + 2\pi\rho$, in other words, it is compactified on a circle of radius ρ . The basic fields are the electro-magnetic operators given by vertex operators $V_{\lambda, \bar{\lambda}} = :\exp(i\lambda x(z) + i\bar{\lambda}\bar{x}(\bar{z})):$ such that they are invariant under a shift $X \rightarrow X + 2\pi\rho$ and that X has a discontinuity

of $2\pi\rho m$ around a vertex operator $V_{\lambda,\bar{\lambda}}$. This implies that the left and right $U(1)$ -charges are quantised as

$$\lambda = \frac{1}{\sqrt{2}} \left(\frac{n}{\rho} + m\rho \right), \quad \bar{\lambda} = \frac{1}{\sqrt{2}} \left(\frac{n}{\rho} - m\rho \right),$$

with integers m and n . All other fields are obtained by multiplying with derivatives of the scalar field. The states are given by Fock spaces $\mathcal{F}_{\lambda,\bar{\lambda}}$ with $V_{\lambda,\bar{\lambda}}$ corresponding to the momentum ground states.

Setting now $\rho = N\sqrt{2}$ for the \mathcal{C}_{2N} model we obtain $4N^2$ sectors \mathcal{H}_j each being a sum of Fock spaces $\mathcal{F}_{\lambda,\bar{\lambda}}$ with left and right $U(1)$ -charges

$$\lambda = 2Nn + \frac{j}{2N}, \quad \bar{\lambda} = 2N\bar{n} + \frac{j}{2N}, \quad n, \bar{n} \in \mathbf{Z}. \quad (30)$$

Each sector \mathcal{H}_j contributes a term $|\eta(\tau)^{-1}\Theta_{j-N^2,2N^2}(\tau)|^2$ to the partition function (17). The set of charges (30) is the same as for the \mathcal{C}_{2N} twisted (ξ, η) system. We thus have a one-to-one correspondence of the \mathcal{C}_{2N} model and the Coulomb gas model at radius ρ and central charge $c = -2$. This correspondence is not just at the level of correlators and the partition function but extends to the operator formalism as well.

Due to the conserved $U(1)$ -currents J, \bar{J} the $U(1)$ -charges behave additively under operator products,

$$V_{\lambda,\bar{\lambda}}(z, \bar{z})\mathcal{F}_{\mu,\bar{\mu}} \subset \mathcal{F}_{\lambda+\mu, \bar{\lambda}+\bar{\mu}}.$$

Hence, the fusion rules respect the charges, $\mathcal{H}_j \times \mathcal{H}_k = \mathcal{H}_{j+k}$. But because of the Feigin-Fuchs form of the stress tensor the $U(1)$ -current is not a Virasoro primary field,

$$T(z)J(w) = \frac{-1}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w} + O(1). \quad (31)$$

As a consequence the system has a charge asymmetry, $J_0^\dagger = 1 - J_0$. This can be seen by taking the adjoint of the $[T, J]$ commutator [9] or by looking for a definition of adjoint for J leaving the Feigin-Fuchs form of the stress-tensor invariant [12]. The charge asymmetry implies anomalous charge conservation in correlation functions,

$$\left\langle V_{\lambda_1, \bar{\lambda}_1}(z_1, \bar{z}_1) \cdots V_{\lambda_n, \bar{\lambda}_n}(z_n, \bar{z}_n) \right\rangle = 0, \quad \text{unless } \Sigma \lambda_i = \Sigma \bar{\lambda}_i = 1. \quad (32)$$

Translated to the fermionic formulation this means that correlators are non-vanishing only if they contain exactly one unpaired ξ and $\bar{\xi}$ field. In particular, we have two degenerate vacua, the Möbius invariant vacuum Ω and $\xi_0 \bar{\xi}_0 \Omega$ such that $\langle \Omega | \xi_0 \bar{\xi}_0 | \Omega \rangle = 1$. As a further consequence of the charge asymmetry it is not possible to have an inner product (\cdot, \cdot) on the vacuum sector compatible with $L_n^\dagger = L_{-n}$. In fact, (v, w) can be non-vanishing only if $v \in \mathcal{H}_j$ and $w \in \mathcal{H}_{2N-j}$. A further problem is the appearance of fields which are neither primary fields nor descendants of a primary field. The simplest example is the current J which gets mapped into the conformal family of the identity by the action of T . The representation of the Virasoro algebra generated from the current J is thus a reducible but indecomposable representation.

The lack of an inner product and the appearance of reducible representations of the Virasoro algebra is a result of the charge-asymmetry and does not occur in Virasoro minimal models. In the Coulomb gas construction of Virasoro minimal models one therefore introduces screening charges which change the $U(1)$ -charge but commute with the Virasoro algebra. The physical states of the minimal model are defined through a BRST resolution [12]. This identifies the two ground states and removes the reducible representations of the Virasoro algebra leaving only the field content of the minimal model. In our case the screening charge corresponds to η_0 and the usual “physical” space $\ker \eta_0 / \text{im } \eta_0$ is trivial, as observed in [8]. We can, however, take the kernel of η_0 without going to the quotient. This defines a “small” algebra which we discuss in the next section.

3 The “small” algebra

It was noted in [9] that the (ξ, η) system contains a “small” algebra generated by $\partial\xi$ and η . In this section we show that this “small” algebra can be characterised as the kernel of η_0 on the space of states for the (ξ, η) system. It has a unique ground state and is completely reducible into Virasoro highest weight representations.

Let us return to the chiral untwisted (ξ, η) system. Its space of states is spanned by lexicographically ordered monomials of the creation modes $\xi_m, m \geq 0$ and $\eta_m, m > 0$ acting on the Möbius invariant vacuum Ω . The operator η_0 vanishes on all states not containing the mode ξ_0 since η_0 and ξ_0 are conjugate operators. The correspondence of states and fields is

$$\xi_{-m_1} \cdots \xi_{-m_r} \eta_{-n_1-1} \cdots \eta_{-n_s-1} \Omega \longleftrightarrow : \frac{\partial^{m_1} \xi(z)}{m_1!} \cdots \frac{\partial^{m_r} \xi(z)}{m_r!} \frac{\partial^{n_1} \eta(z)}{n_1!} \cdots \frac{\partial^{n_s} \eta(z)}{n_s!} :.$$

The zero mode ξ_0 can only be generated by the field $\xi(z)$ and is not present in derivatives of $\xi(z)$. Hence, if we start off with states not containing ξ_0 the zero mode does not get generated. We can therefore consistently restrict to the kernel of η_0 . This space is spanned by ordered monomials of the negative modes $\xi_m, \eta_m, m < 0$ acting on the vacuum Ω . All the fields can be generated by taking derivatives and normal ordered products of $\partial\xi$ and η . This is the “small” algebra of [9]. For any state v in the “small” algebra there is a second state $\xi_0 v$ in the (ξ, η) system.

Both $\partial\xi$ and η are Virasoro primary fields of dimension one and have $U(1)$ -charges ± 1 . We can put them on equal footing by writing them as the two components of a fermionic dimension one field ψ ,

$$\psi^+(z) = \eta(z), \quad \psi^-(z) = \partial\xi(z). \quad (33)$$

The anti-commutator (4) reads then

$$\{\psi_m^\alpha, \psi_n^\beta\} = m J^{\alpha\beta} \delta_{m+n}, \quad (34)$$

where the anti-symmetric tensor J is defined as $J^{+-} = -J^{-+} = 1$. For the stress tensor we find

$$T(z) = \frac{1}{2} J_{\alpha\beta} : \psi^\alpha(z) \psi^\beta(z) :, \quad (35)$$

where $J_{\alpha\beta}$ is the inverse of $J^{\alpha\beta}$. The space of states of the “small” algebra is given by ordered monomials of negative modes of ψ^+ and ψ^- acting on the vacuum. Because of Fermi statistics each mode appears at most once.

Correlation functions of the ψ field can easily be calculated using a fermionic version of Wick’s theorem and are given by the Pfaffian

$$\langle \psi^{\alpha_1}(z_1) \cdots \psi^{\alpha_{2n}}(z_{2n}) \rangle = \text{Pf} \left(\frac{J^{\alpha_i \alpha_j}}{(z_i - z_j)^2} \right). \quad (36)$$

The “small” algebra is determined completely by the anti-commutators (34) of the generating fermion fields and the expression (35) for the stress tensor. It has a global $SL(2)$ symmetry, which we will call isospin, acting on the basic fermion field ψ according to

$$g: \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} \mapsto \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}$$

with the matrix g_{ij} in $SL(2)$. We can choose an explicit basis of the Lie algebra $sl(2)$, acting on the basic fermion field ψ as

$$J^0 \psi^\pm = \pm \frac{1}{2} \psi^\pm, \quad J^\pm \psi^\pm = 0, \quad J^\pm \psi^\mp = \psi^\pm, \quad (37)$$

and on their products in the usual way. The basic fermion field ψ thus transforms as the isospin $\frac{1}{2}$ representation of $sl(2)$. The original $U(1)$ -symmetry of the (ξ, η) system is the $U(1)$ -subgroup generated by J^0 .

The space of states will decompose into representations of $SL(2) \times \text{Vir}$. The Virasoro primary states in the isospin j multiplet can be obtained explicitly as

$$\phi^{j,m} = \psi_{-2j}^{(+)} \cdots \psi_{-j+m}^{+} \psi_{-j+m+1}^{-} \cdots \psi_{-1}^{-} \Omega, \quad (38)$$

where $j \in \frac{1}{2}\mathbf{Z}$, $m = -j, -j+1, \dots, j$ and we symmetrise over the signs. The generating fermions ψ^\pm form the isospin $\frac{1}{2}$ multiplet. The structure constants for the $\phi^{j,m}$ fields are invariant under $SL(2)$, that is, they are Clebsch-Gordan coefficients up to some normalisation factor.

The fields with half-integral isospin are fermionic while the fields with integral isospin are bosonic. The bosonic sector is generated by the isospin 1 fields,

$$W^+ = \phi^{1,1} = \partial \psi^+ \psi^+, \quad (39)$$

$$W^0 = \phi^{1,0} = \frac{1}{2} (\partial \psi^+ \psi^- + \partial \psi^- \psi^+), \quad (40)$$

$$W^- = \phi^{1,-1} = \partial \psi^- \psi^-. \quad (41)$$

As they are of dimension three they do not form a Kac-Moody algebra but a W-algebra [13] with operator product expansion

$$T(z)T(w) \sim \frac{-1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \quad (42)$$

$$T(z)W^j(w) \sim \frac{3W^j(w)}{(z-w)^2} + \frac{\partial W^j(w)}{z-w}, \quad (43)$$

$$\begin{aligned} W^i(z)W^j(w) \sim & g^{ij} \left(\frac{1}{(z-w)^6} - 3 \frac{T(w)}{(z-w)^4} + \frac{3}{2} \frac{\partial T(w)}{(z-w)^3} + \right. \\ & + \frac{3}{2} \frac{\partial^2 T(w)}{(z-w)^2} - 4 \frac{(T^2)(w)}{(z-w)^2} + \frac{1}{6} \frac{\partial^3 T(w)}{z-w} - 4 \frac{\partial(T^2)(w)}{z-w} \Big) \\ & - 5f_k^{ij} \left(\frac{W^k(w)}{(z-w)^3} + \frac{1}{2} \frac{\partial W^k(w)}{(z-w)^2} + \frac{1}{25} \frac{\partial^2 W^k(w)}{z-w} + \frac{1}{25} \frac{(TW^k)(w)}{z-w} \right), \end{aligned} \quad (44)$$

where g^{ij} is the metric on the isospin one representation, $g^{+-} = g^{-+} = 2$, $g^{00} = -1$, and f_k^{ij} are the structure constants of $sl(2)$, such that $f^{ijk} = f_l^{ij} g^{lk}$ is totally antisymmetric and normalised to $f^{+-0} = 2$.

Each isospin j multiplet appears exactly once in the “small” algebra as can be seen by introducing the $sl(2) \times \text{Vir}$ character

$$\chi(\tau, z) = \text{tr} \left(w^{J^0} q^{L_0 - \frac{c}{24}} \right), \quad (45)$$

where $w = \exp(2\pi iz)$. The fields $\phi^{j,m}$ have conformal weight $j(2j+1)$ which corresponds to $h_{2j+1,1}$ in the Kac-table for $c = -2$. The Virasoro character for each of the $\phi^{j,m}$ is thus

$$\chi_{2j+1,1}^{\text{Vir}}(\tau) = q^{1/8} \eta(\tau)^{-1} \left(q^{j(2j+1)} - q^{(j+1)(2j+1)} \right)$$

The sum over the isospin j multiplets is then

$$\begin{aligned} \chi(\tau, z) &= \sum_{j \in \frac{1}{2}\mathbf{Z}_{\geq 0}} \sum_{k=-j}^j w^k \chi_{2j+1,1}^{\text{Vir}}(\tau) \\ &= \eta(\tau)^{-1} \sum_{l=-\infty}^{\infty} \frac{w^{l/2} - w^{-l/2}}{w^{1/2} - w^{-1/2}} q^{(2l-1)^2/8}. \end{aligned} \quad (46)$$

This agrees with the direct calculation

$$\chi(\tau, w) = q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1 + w^{\frac{1}{2}} q^n) (1 + w^{-\frac{1}{2}} q^n). \quad (47)$$

The Ramond sector, with ground state energy $h = -1/8$, decomposes in the same way into isospin multiplets with each multiplet appearing exactly once. The Virasoro primary states $\chi^{j,m}$ in the isospin j multiplet of the Ramond sector can be obtained explicitly as

$$\chi^{j,m} = \psi_{-2j+1/2}^{(+)} \cdots \psi_{-j+m+1/2}^{+} \psi_{-j+m+3/2}^{-} \cdots \psi_{-1/2}^{-} \sigma_{1/2}, \quad (48)$$

where $j \in \frac{1}{2}\mathbf{Z}$, $m = -j, -j+1, \dots, j$ and we symmetrise over the signs. Due to the half-integral moding, $\chi^{j,m}$ has conformal weight $2j^2 - 1/8$, which corresponds to $h_{2j+1,2}$ in the Kac-table. Their Virasoro characters are

$$\chi_{2j+1,2}^{\text{Vir}}(\tau) = \eta(\tau)^{-1} \left(q^{2j^2} - q^{2(j+1)^2} \right)$$

leading to the $sl(2) \times \text{Vir}$ character for the Ramond sector

$$\begin{aligned}
\chi^R(\tau, z) &= \sum_{j \in \frac{1}{2}\mathbf{Z}_{\geq 0}} \sum_{k=-j}^j w^k \chi_{2j+1,2}^{\text{Vir}}(\tau) \\
&= \eta(\tau)^{-1} \sum_{l=-\infty}^{\infty} w^{l/2} q^{l^2/2} \\
&= q^{-\frac{1}{24}} \prod_{n=1}^{\infty} (1 + w^{\frac{1}{2}} q^{n+\frac{1}{2}})(1 + w^{-\frac{1}{2}} q^{n+\frac{1}{2}}).
\end{aligned} \tag{49}$$

Starting with the “small” algebra one can repeat the orbifold constructions of the previous section. The characters of the twisted sectors will be unchanged. However, the character (46) of the untwisted sector has no nice modular transformation properties. Take for example the \mathcal{C}_2 orbifold. The characters of the four (chiral) sectors are

$$\begin{aligned}
\chi_0(\tau) &= \frac{1}{2} \left(\frac{\Theta_{1,2}(\tau)}{\eta(\tau)} + \eta(\tau)^2 \right), & \chi_1(\tau) &= \eta(\tau)^{-1} \Theta_{0,2}(\tau), \\
\chi_2(\tau) &= \frac{1}{2} \left(\frac{\Theta_{1,2}(\tau)}{\eta(\tau)} - \eta(\tau)^2 \right), & \chi_3(\tau) &= \eta(\tau)^{-1} \Theta_{2,2}(\tau).
\end{aligned} \tag{50}$$

The partition function should be the diagonal combination of these characters. But the characters χ_0 and χ_2 are sums of a modular function and a modular form of weight one. Defining formally an action of the modular group on these characters results in characters having an additional prefactor of $\log q = 2\pi i\tau$,

$$\chi_0(-1/\tau) = \frac{1}{4} \chi_1(\tau) - \frac{1}{4} \chi_3(\tau) - \frac{i\tau}{2} \eta(\tau)^2, \tag{51}$$

$$\chi_1(-1/\tau) = \chi_0(\tau) + \frac{1}{2} \chi_1(\tau) + \chi_2(\tau) + \frac{1}{2} \chi_3(\tau), \tag{52}$$

$$\chi_2(-1/\tau) = \frac{1}{4} \chi_1(\tau) - \frac{1}{4} \chi_3(\tau) + \frac{i\tau}{2} \eta(\tau)^2, \tag{53}$$

$$\chi_3(-1/\tau) = -\chi_0(\tau) + \frac{1}{2} \chi_1(\tau) - \chi_2(\tau) + \frac{1}{2} \chi_3(\tau). \tag{54}$$

The situation for a general \mathcal{C}_{2N} orbifold is analogous. The twisted sectors are the same as for the (ξ, η) -system while the untwisted sector splits into linear combinations of $\eta(\tau)^2$ and terms involving theta functions, $\eta(\tau)^{-1} \Theta_{(2k-N)N, 2N^2}(\tau)$ for $k = 0, \dots, 2N - 1$. For a discussion of these issues and an attempt at a physical interpretation and construction of modular invariant partition functions see [14].

3.1 Bosonisation and fusion

The \mathcal{C}_2 -twisted (ξ, η) system and “small” algebra share many features with minimal models: The central charge $c = -2$ can be obtained from the minimal model formula

$$c = 1 - 6 \frac{(p - p')^2}{pp'} \tag{55}$$

for $(p, p') = (1, 2)$. The conformal weights can all be found in the Kac-table as

$$h_{n,n'} = \frac{(2n - n')^2 - 1}{8} \quad (56)$$

corresponding to the $U(1)$ -charges $\lambda_{n,n'} = -n + (n' + 1)/2$. The \mathcal{C}_{2N} model for higher twists, $N > 1$, can not be interpreted in minimal model language since they involve fields with non-integral labels in the Kac-table. However, neither \mathcal{C}_2 model should strictly be called a $(1, 2)$ minimal model for the following reasons. Firstly, both models contain an infinite number of Virasoro primary fields in contrast to the usual minimal models. Secondly, the treatment of minimal models following Belavin, Polyakov and Zamolodchikov [1] supposes a specific embedding structure of Virasoro modules and Fock spaces which is only the case for $p, p' > 1$ and relatively prime [15]. And thirdly, both models contain fields which belong to reducible but indecomposable representations of the Virasoro algebra.

In the (ξ, η) system the operator product expansions of the Ramond-fields can easily be calculated in the bosonised formalism and one obtains, for example,

$$\chi^{0,0}(z)\chi^{0,0}(w) \sim (z-w)^{\frac{1}{4}} \left\{ \xi(w) + \frac{1}{2}\psi^-(w)(z-w) + \dots \right\} \quad (57)$$

$$\begin{aligned} \chi^{0,0}(z)\chi^{\frac{1}{2},\frac{1}{2}}(w) &\sim (z-w)^{-\frac{1}{4}} \left\{ \mathbf{1} + \frac{1}{2}J(w)(z-w) \right. \\ &\quad + \left(\frac{1}{4}T(w) + \frac{1}{8}\partial J(w) \right) (z-w)^2 \\ &\quad \left. + \left(\frac{5}{48}\partial T(w) + \frac{1}{32}\partial^2 J(w) + \frac{1}{24}W^0(w) \right) (z-w)^3 + \dots \right\} \end{aligned} \quad (58)$$

On the right hand side we find the conformal families generated by the Virasoro primary fields of the identity, $\phi^{0,0} = \mathbf{1}$, the isospin 1/2 field $\phi^{1/2,-1/2} = \psi^- = \partial\xi$ and the isospin 1 field $\phi^{1,0} = W^0$. But we also produce the fields ξ and J generating reducible representations, such that

$$L_1 J = -\Omega, \quad \left(L_{-2} - \frac{1}{2}L_{-1}^2 \right) J = W^0, \quad L_{-1}\xi = \psi^- \quad (59)$$

on the states. The fact that the identity and J are coupled in one reducible representation makes it possible for the identity to appear in the operator product of two primary fields of different conformal weight.

Correlation functions for the “small” algebra can be calculated from null vector constraints: The states $\chi^{j,m}$ have null vectors at level $2(2j + 1)$ and as a consequence the 4pt function $\langle \chi^{j_1,m_1}(z_1) \dots \chi^{j_4,m_4}(z_4) \rangle$ satisfies a Fuchsian differential equation. For the full, non-chiral 4pt function we have to take linear combinations of the chiral and anti-chiral solutions to the differential equations such that the full correlator has no monodromy. The generic solution of the differential equation and monodromy conditions is given by the integral representation of Dotsenko and Fateev [16]. Consider now specifically the 4pt function

for the ground state of the Ramond sector, $\mu(z, \bar{z}) = \chi^{0,0}(z)\bar{\chi}^{0,0}(\bar{z})$. The 4pt correlator $\langle \mu\mu\mu\mu \rangle$ calculated according to [16] vanishes identically. This can be traced back to the Fuchsian differential equation satisfied by the correlators acquiring exponents differing by an integer. Hence some solutions of the differential equation become degenerate. The linear combination of these degenerate solutions appearing in the correlation function is such that they cancel identically. The generic solution of [16] is only valid in the non-degenerate case. To get a non-zero result one has to consider the $c \rightarrow -2$ limit and obtains [8]

$$\langle \mu\mu\mu\mu \rangle = |z_{12}z_{34}|^{\frac{1}{2}} |x(1-x)|^{\frac{1}{2}} \left(F(x)\overline{F(1-x)} + F(1-x)\overline{F(x)} \right), \quad (60)$$

where $x = (z_{12}z_{34})/(z_{13}z_{24})$ is the cross-ratio and $F(x)$ is the hypergeometric function

$$F(x) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right).$$

This result can be obtained without taking limits in the \mathcal{C}_2 twisted “small” algebra using the techniques of [17] or by directly solving the differential equation and monodromy conditions at $c = -2$ [10].

Conformal invariance fixes 4pt functions up to a function of the cross-ratios x, \bar{x} . This function can be written as a sum over products of a chiral conformal block times an anti-chiral conformal block corresponding to the various fields propagating in the intermediate channel. If we keep z_1 near z_2 and z_3 near z_4 and pull the two pairs apart, corresponding to the limit $x \rightarrow 0$, the only fields appearing in the intermediate channel of a conformal block for $\langle \mu(z_1) \cdots \mu(z_4) \rangle$ are those which appear in the fusion of $\chi^{0,0}$ with itself. The 4pt function (60) has logarithmic singularities at $x = 0$ and $x = 1$. Analytic continuation of the hypergeometric function yields

$$F(1-x) = -\frac{1}{\pi} \left(\ln(x/16)F(x) + M(x) \right),$$

where $M(x)$ is some regular function vanishing at the origin. We thus have

$$\langle \mu\mu\mu\mu \rangle = |z_{12}z_{34}|^{\frac{1}{2}} |x(1-x)|^{\frac{1}{2}} \left\{ C_1 |F(x)|^2 + C_2 \left((\ln(x)F(x) + M(x))\overline{F(x)} + c.c. \right) \right\}, \quad (61)$$

with constants C_1, C_2 . Conformal blocks can be calculated perturbatively with the techniques of Appendix B of [1]. The first summand in (61) can indeed be identified as the conformal block for coupling through the vacuum. However, the second summand has a contribution with a $\ln(x)$ behaviour for small values of the cross-ratio x . It was argued in [10] that this implies the appearance of “logarithmic” operators in the operator product expansion $\chi^{0,0}(z)\chi^{0,0}(0)$. These operators correspond to 2-dimensional Jordan cells for L_0 and \bar{L}_0 . The perturbative calculation of conformal blocks can be adapted to such “logarithmic” operators and reproduces exactly the second summand in (61) if one couples through a field $\omega(z, \bar{z})$ with operator product expansions,

$$T(z)\omega(w, \bar{w}) \sim \frac{1}{(z-w)^2} + \frac{\partial\omega(w, \bar{w})}{z-w}, \quad (62)$$

$$\bar{T}(\bar{z})\omega(w, \bar{w}) \sim \frac{1}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\omega(w, \bar{w})}{\bar{z}-\bar{w}}. \quad (63)$$

Such a field does not exist in the twisted (ξ, η) system nor the twisted “small” algebra since there L_0 is always diagonalisable. While the twisted (ξ, η) system has well-defined fusion albeit involving reducible representations of the Virasoro algebra, the twisted “small” algebra is not closed under fusion. To define a consistent conformal field theory one has to extend the space of states to include two-dimensional Jordan cells for L_0 and \bar{L}_0 . Let us therefore consider a representation of the chiral “small” algebra containing it. We assume L_0 has a lowest eigenvalue on the representation such that the energy is bounded from below. On the lowest energy subspace all positive modes of ψ^\pm vanish and we have $L_0 = \psi_0^- \psi_0^+$ because of the normal ordering prescription. The zero modes of ψ^\pm form a two-dimensional Grassmann algebra. The lowest energy subspace will decompose into representations of that Grassmann algebra. Its maximal indecomposable representation is four-dimensional, spanned by $\{\Omega, \phi^\pm, \omega\}$ such that

$$\psi_0^\mp \phi^\pm = \pm i\Omega, \quad \psi_0^\pm \omega = i\phi^\pm, \quad L_0 \omega = \Omega, \quad (64)$$

and the other actions vanish. The factors of i are chosen to agree with the conventions of the following section. From (64) we see that L_0 has zero eigenvalue on the lowest energy subspace with (ω, Ω) forming a two-dimensional Jordan cell. The full representation is generated from (64) by the free action of the negative modes of ψ^\pm . For any state built on the Möbius invariant vacuum state Ω , corresponding to a state of the “small” algebra, there are three other states in the representation generated from (64) built on ϕ^\pm and ω . The maximal extension of the chiral “small” algebra has thus four ground states, two bosonic states $\{\Omega, \omega\}$ in a Jordan block and two fermion states ϕ^\pm of conformal weight zero. In a non-chiral model we can have at most 16 ground states corresponding to the product of the left times the right maximal extension.

4 Symplectic fermions

In this section we will present a construction for the maximal diagonal extension of the non-chiral “small” algebra based on “symplectic” fermions. Specifically we consider a free fermionic field $\Phi(z, \bar{z})$ taking values in a two-dimensional space with symplectic form $J^{\alpha\beta}$. The action is

$$S = \frac{1}{4\pi} \int d^2z J_{\alpha\beta} \partial_\mu \Phi^\alpha \partial^\mu \Phi^\beta, \quad (65)$$

where $J_{\alpha\beta}$ is the inverse of the symplectic form, $J_{\alpha\beta} J^{\beta\gamma} = \delta_\alpha^\gamma$. The propagator is

$$\langle \Phi^\alpha(z, \bar{z}) \Phi^\beta(z', \bar{z}') \rangle = -J^{\alpha\beta} \ln |z - z'|^2. \quad (66)$$

The general solution of the equations of motion is

$$\Phi^\alpha(z, \bar{z}) = \phi^\alpha(z) + \bar{\phi}^\alpha(\bar{z}), \quad (67)$$

where ϕ^α and $\bar{\phi}^\alpha$ are arbitrary functions of their respective arguments, subject only to periodicity or boundary conditions. Invariance of the action under $\sigma \mapsto \sigma + 2\pi$ implies the

mode expansion

$$\phi^\alpha(z) = \phi_0^\alpha - i\psi_0^\alpha \ln z + i \sum_{n \neq 0} \frac{\psi_n^\alpha}{n} z^{-n}, \quad (68)$$

and analogously for $\bar{\phi}^\alpha(\bar{z})$. The chiral fields $\phi^\alpha, \bar{\phi}^\alpha$ are not completely independent but are coupled through their zero-modes. The periodicity condition $\Phi^\alpha(e^{2\pi i} z, e^{-2\pi i} \bar{z}) = \Phi^\alpha(z, \bar{z})$ implies $\psi_0^\alpha = \bar{\psi}_0^\alpha$ on the physical states. Thus, the zero-mode algebra acting on physical states involves only the combinations

$$\Phi_0^\alpha = \phi_0^\alpha + \bar{\phi}_0^\alpha, \quad \Psi_0^\alpha = \frac{1}{2}(\psi_0^\alpha + \bar{\psi}_0^\alpha). \quad (69)$$

The mode expansion for the non-chiral field Φ^α can then be written

$$\Phi^\alpha(z, \bar{z}) = \Phi_0^\alpha - i\Psi_0^\alpha \ln |z|^2 + i \sum_{n \neq 0} \frac{\psi_n^\alpha}{n} z^{-n} + \frac{\bar{\psi}_n^\alpha}{n} \bar{z}^{-n}. \quad (70)$$

The modes $\phi_0^\alpha, \psi_n^\alpha$ for $n < 0$ are creation modes while ψ_n^α for $n \geq 0$ are annihilation modes. We use the fermionic normal ordering prescription

$$:\phi_0^\alpha \psi_0^\beta: = -:\psi_0^\beta \phi_0^\alpha: = \phi_0^\alpha \psi_0^\beta, \quad :\psi_m^\alpha \psi_n^\beta: = \begin{cases} \psi_m^\alpha \psi_n^\beta & \text{for } m < 0, \\ -\psi_n^\beta \psi_m^\alpha & \text{for } m > 0. \end{cases} \quad (71)$$

The propagator (66) then implies the anti-commutators

$$\{\psi_m^\alpha, \psi_n^\beta\} = m J^{\alpha\beta} \delta_{m+n}, \quad \{\phi_0^\alpha, \psi_0^\beta\} = \{\Phi_0^\alpha, \Psi_0^\beta\} = i J^{\alpha\beta}. \quad (72)$$

The logarithmic propagator shows that correlators involving Φ require a careful treatment of the infra-red and ultra-violet divergences. However, derivatives of Φ are without problems and reproduce the results of the previous section. The action is invariant under a constant shift of the fermion fields, $\delta\Phi^\alpha(z, \bar{z}) = \beta^\alpha$, resulting in the conserved currents

$$\psi^\alpha(z) = i\partial\Phi^\alpha(z, \bar{z}), \quad \bar{\psi}^\alpha(\bar{z}) = i\bar{\partial}\Phi^\alpha(z, \bar{z}), \quad (73)$$

which are the dimension one fermions of the previous section. Their propagators are

$$\begin{aligned} \langle \psi^\alpha(z) \psi^\beta(w) \rangle &= \frac{J^{\alpha\beta}}{(z-w)^2}, & \langle \Phi^\alpha(z, \bar{z}) \psi^\beta(w) \rangle &= i J^{\alpha\beta} \frac{1}{z-w}, \\ \langle \bar{\psi}^\alpha(z) \bar{\psi}^\beta(w) \rangle &= \frac{J^{\alpha\beta}}{(\bar{z}-\bar{w})^2}, & \langle \Phi^\alpha(z, \bar{z}) \bar{\psi}^\beta(\bar{w}) \rangle &= i J^{\alpha\beta} \frac{1}{\bar{z}-\bar{w}}, \\ \langle \psi^\alpha(z) \bar{\psi}^\beta(\bar{w}) \rangle &= 0, \end{aligned} \quad (74)$$

showing that the “small” algebra is contained in the symplectic fermion model. The stress tensor obtained from the action is

$$T_{\mu\nu} = -\frac{1}{2} J_{\alpha\beta} \partial_\mu \Phi^\alpha \partial_\nu \Phi^\beta + \frac{1}{4} g_{\mu\nu} J_{\alpha\beta} \partial_\lambda \Phi^\alpha \partial^\lambda \Phi^\beta. \quad (75)$$

Its components in the complex basis are

$$T(z) = \frac{1}{2} J_{\alpha\beta} :\psi^\alpha(z) \psi^\beta(z):, \quad \bar{T}(\bar{z}) = \frac{1}{2} J_{\alpha\beta} :\bar{\psi}^\alpha(\bar{z}) \bar{\psi}^\beta(\bar{z}): \quad (76)$$

and the central charge is $c = -2$.

The vacuum representation is generated from the vacuum Ω by the action of the creation modes Φ_0^α and $\psi_{-n}^\alpha, \bar{\psi}_{-n}^\alpha$ for $n > 0$. The partition function is thus $D_{0,0}(\tau)$ as for the (ξ, η) -system. In particular, there are four states of dimension zero, the two fermionic states $\Phi_0^\alpha \Omega$, the vacuum state Ω and another bosonic state

$$\omega = \frac{1}{2} J_{\alpha\beta} \Phi_0^\alpha \Phi_0^\beta \Omega. \quad (77)$$

The fermionic states $\Phi_0^\alpha \Omega$ are eigenstates of L_0 and \bar{L}_0 while the vacuum and ω form a two-dimensional Jordan cell to the eigenvalue zero for L_0 ,

$$L_0 \omega = \bar{L}_0 \omega = \Omega$$

From this one finds the operator product expansion of the field $\omega(z, \bar{z})$ with the stress tensor as in (62) and (63) indicating that $\omega(z, \bar{z})$ does indeed provide the “logarithmic” operator required in the 4pt functions of twist fields [10].

The symplectic fermions provide a non-chiral version of the maximal extension of the “small” algebra discussed in the previous section. It is non-chiral since the left and right zero modes are coupled through the periodicity condition. The (ξ, η) system and the symplectic fermions both have four ground states and are two different subtheories of the product theory generated by the “chiral” fields $\phi^\alpha(z)$ and the “anti-chiral” fields $\bar{\phi}^\alpha(\bar{z})$. This product theory is rather problematic, however, since the “chiral” fields are not meromorphic functions of the coordinates z but have a $\log(z)$ dependence as well. Specifically, setting $\psi_0^- \equiv 0$ and $\bar{\psi}_0^- \equiv 0$ removes their partners $\phi_0^+, \bar{\phi}_0^+$ from the theory and we obtain the (ξ, η) system with the identification $\xi = \phi^-, \eta = i\partial\phi^+$ and $\bar{\xi} = \bar{\phi}^-, \bar{\eta} = i\bar{\partial}\bar{\phi}^+$. The field ξ is then a proper chiral field since the troublesome $\log(z)$ term has been removed by the $\psi_0^\pm \equiv 0$ condition. If instead we take the diagonal choice, $\psi_0^\pm \equiv \bar{\psi}_0^\pm$, we obtain the symplectic fermions. Setting both $\psi_0^\pm \equiv 0$ and $\bar{\psi}_0^\pm \equiv 0$ yields the “small” algebra.

4.1 $SL(2)$ symmetry

The action (65) is invariant under $SL(2)$ transformations on the field Φ . We write an infinitesimal transformation as

$$\delta\Phi^\alpha(z, \bar{z}) = -i\Lambda_a(z, \bar{z}) d_\beta^{a\alpha} \Phi^\beta(z, \bar{z}), \quad (78)$$

where the $d_\beta^{a\alpha}$ are representation matrices for the two-dimensional representation of $sl(2)$, see Appendix B for conventions used. Noether’s theorem then leads to currents

$$J^a(z, \bar{z}) = \frac{i}{2} d_{\alpha\beta}^a : \Phi^\alpha(z, \bar{z}) \psi^\beta(z) :, \quad \bar{J}^a(z, \bar{z}) = \frac{i}{2} d_{\alpha\beta}^a : \Phi^\alpha(z, \bar{z}) \bar{\psi}^\beta(\bar{z}) :, \quad (79)$$

where we wrote explicitly the dependence on the holomorphic and anti-holomorphic coordinate and $d_{\alpha\beta}^a = d_{\alpha\beta}^{a\gamma} J_{\gamma\beta}$. The currents are conserved,

$$\bar{\partial} J^a + \partial \bar{J}^a = 0. \quad (80)$$

Since the currents contain the field Φ directly they require careful renormalisation beyond the normal ordering prescription (71). However the charges are well-defined,

$$\begin{aligned} Q^a &= \frac{1}{2\pi i} \int (J^a dz - \bar{J}^a d\bar{z}) \\ &= d_{\alpha\beta}^a \left\{ i\Phi_0^\alpha \Psi_0^\beta + \sum_{n=1}^{\infty} \left(\frac{\psi_{-n}^\alpha \psi_n^\beta}{n} + \frac{\bar{\psi}_{-n}^\alpha \bar{\psi}_n^\beta}{n} \right) \right\}. \end{aligned}$$

and satisfy an $sl(2)$ algebra,

$$[Q^a, Q^b] = f_c^{ab} Q^c. \quad (81)$$

One can check explicitly that the fermion field Φ transforms under the two-dimensional representation of $SL(2)$,

$$[Q^a, \Phi^\alpha(z, \bar{z})] = d_\beta^{a\alpha} \Phi^\beta(z, \bar{z}), \quad (82)$$

and that the fundamental propagator (66) is invariant under the $SL(2)$ transformations generated by Q^a . Thus, the $SL(2)$ symmetry of symplectic fermion model (65) remains valid in the quantum theory.

4.2 Orbifolds

Given a modular invariant conformal field theory with a (finite) symmetry group Γ one can construct another such theory as the orbifold by Γ [18, 19]. In the Hamiltonian picture the orbifold theory is obtained by adding twisted sectors, corresponding to field configurations which close along the “space” cycle of the torus only up to an element $h \in \Gamma$, and then projecting onto group invariant states. In the Lagrangian picture this corresponds to a sum over partition functions with different boundary conditions: For $g, h \in \Gamma$ we denote the partition function of the h -twisted sector with an insertion of the operator g as

$$g \square_h = \text{tr}_{\mathcal{M}_h} \left(g q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \right). \quad (83)$$

The Γ -orbifold partition function is then obtained as

$$Z[\Gamma] = \frac{1}{|\Gamma|} \sum_{\substack{g, h \in \Gamma \\ gh = hg}} g \square_h. \quad (84)$$

If Γ is non-abelian, boundary conditions twisted by non-commuting group elements are not consistent, hence the condition $gh = hg$ [20].

In the case at hand we are interested in orbifolds of the symplectic fermion model (65) by finite subgroups of $SL(2)$. These are the binary cyclic groups \mathcal{C}_N , the binary dihedral groups \mathcal{D}_N and the binary tetrahedral, octahedral and icosahedral group $\mathcal{T}, \mathcal{O}, \mathcal{I}$. We will not pursue the \mathcal{C}_N orbifolds for N odd since they contain fermions and the modular invariant partition function resulting from sum over spin structures is the same as that for the \mathcal{C}_{2N} -orbifold.

Consider first the (abelian) binary cyclic group of even order \mathcal{C}_{2N} . We choose a torus T containing the cyclic group, $\mathcal{C}_{2N} \subset T \subset SL(2)$. The generator of the torus can be written as $K_\alpha^\beta = n_a d_\alpha^{a\beta}$, with $n_1^2 + n_2^2 - n_0^2 = 1$, that is, the vector $\mathbf{n} = (n_0, n_1, n_2)$ is a unit vector with respect to the metric η on the group $SL(2)$ (see Appendix B). The cyclic group \mathcal{C}_{2N} is then generated by $h = \exp(2\pi i Q/N)$, where $Q = n_a Q^a$. The fermion field Φ has two components Φ^\pm with eigenvalues $\pm 1/2$ under Q . As in the (ξ, η) system we introduce twisted sectors \mathcal{M}_λ , with boundary conditions

$$\Phi^\pm(e^{2\pi i} z, e^{-2\pi i} \bar{z}) = e^{\pm 2\pi i \lambda} \Phi^\pm(z, \bar{z}). \quad (85)$$

For the \mathcal{C}_{2N} orbifold λ will take the values $\lambda = k/(2N)$ with $k = 1, \dots, 2N - 1$. The boundary conditions imply that Φ^\pm has spin $\pm \lambda$ and can be expanded in modes as

$$\Phi^\pm(z, \bar{z}) = i \sum_{m \in \mathbf{Z}} \left(\frac{\psi_{m \mp \lambda}^\pm}{m \mp \lambda} z^{-m \pm \lambda} + \frac{\bar{\psi}_{m \pm \lambda}^\pm}{m \pm \lambda} \bar{z}^{-m \mp \lambda} \right). \quad (86)$$

The ground state energy in the twisted sector is as before

$$h_\lambda = \bar{h}_\lambda = \frac{\lambda(\lambda - 1)}{2}. \quad (87)$$

We can again introduce the $\mathcal{C}_{2N} \times \text{Vir}$ character

$$\begin{aligned} D_{\mu, \lambda}(\tau) &= \text{tr}_{\mathcal{M}_\lambda} \left(e^{4\pi i \mu Q} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \right) \\ &= (q\bar{q})^{(1-6\lambda(1-\lambda))/12} \prod_{n=1}^{\infty} \left| 1 + e^{2\pi i \mu} q^{n+\lambda-1} \right|^2 \left| 1 + e^{-2\pi i \mu} q^{n-\lambda} \right|^2 \\ &= |d_{\mu, \lambda}(\tau)|^2. \end{aligned} \quad (88)$$

which equals the character for the (ξ, η) system. Hence the partition function is the same as for the \mathcal{C}_{2N} twisted (ξ, η) system,

$$Z[\mathcal{C}_{2N}] = Z_N. \quad (89)$$

To calculate the partition function $Z[\Gamma]$ for non-abelian Γ we follow [6] and add the contributions of the mutually commuting subsets of Γ , which form cyclic groups, subtracting any overcounting. As in [6] the result is

$$Z[\mathcal{D}_N] = \frac{1}{2}(Z_N + 2Z_2 - Z_1), \quad (90)$$

$$Z[\mathcal{T}] = \frac{1}{2}(2Z_3 + Z_2 - Z_1), \quad (91)$$

$$Z[\mathcal{O}] = \frac{1}{2}(Z_4 + Z_3 + Z_2 - Z_1), \quad (92)$$

$$Z[\mathcal{I}] = \frac{1}{2}(Z_5 + Z_3 + Z_2 - Z_1). \quad (93)$$

Here, of course, one has to keep in mind that $Z_n(\tau)$ corresponds to a Coulomb gas partition function at radius $n\sqrt{2}$ and not n as for $c = 1$.

4.3 Marginal operators

Deformations of a conformal field theory, preserving conformal invariance and central charge c , are generated by marginal operators, that is, bosonic operators of conformal weights $(1, 1)$. In the maximal bosonic theory, the \mathcal{C}_2 -model, we have eight bosonic weight $(1, 1)$ operators, four of which are obtained as products of left times right weight one fermions,

$$U(z, \bar{z}) = \frac{1}{2} J_{\alpha\beta} \psi^\alpha(z) \bar{\psi}^\beta(\bar{z}), \quad V^a(z, \bar{z}) = \frac{1}{2} d_{\alpha\beta}^a \psi^\alpha(z) \bar{\psi}^\beta(\bar{z}). \quad (94)$$

The other four are obtained by multiplying with the “logarithmic” field $\omega = \frac{1}{2} J_{\alpha\beta} \Phi^\alpha \Phi^\beta$,

$$\tilde{U}(z, \bar{z}) = \frac{1}{4} J_{\alpha\beta} J_{\gamma\delta} \psi^\alpha(z) \bar{\psi}^\beta(\bar{z}) \Phi^\gamma(z, \bar{z}) \Phi^\delta(z, \bar{z}), \quad (95)$$

$$\tilde{V}^a(z, \bar{z}) = \frac{1}{4} d_{\alpha\beta}^a J_{\gamma\delta} \psi^\alpha(z) \bar{\psi}^\beta(\bar{z}) \Phi^\gamma(z, \bar{z}) \Phi^\delta(z, \bar{z}). \quad (96)$$

The latter operators can also be obtained from the $sl(2)$ currents,

$$:J^a(z, \bar{z}) \bar{J}^b(z, \bar{z}): = -\frac{1}{4} f_c^{ab} \tilde{V}^c(z, \bar{z}) - \frac{1}{4} g^{ab} \tilde{U}(z, \bar{z}). \quad (97)$$

The operators U, \tilde{U} and V^a, \tilde{V}^a transform as singlets and triplets, respectively, under the charge algebra,

$$\begin{aligned} [Q^a, U(z, \bar{z})] &= 0, & [Q^a, V^b(z, \bar{z})] &= f_c^{ab} V^c(z, \bar{z}), \\ [Q^a, \tilde{U}(z, \bar{z})] &= 0, & [Q^a, \tilde{V}^b(z, \bar{z})] &= f_c^{ab} \tilde{V}^c(z, \bar{z}). \end{aligned} \quad (98)$$

These operators have vanishing 3pt functions, an integrability condition which ensures they remain marginal when the perturbation is switched on [21, 22].

The effect of a perturbation by U is a change the normalisation of the action since U is precisely the operator appearing in the action,

$$U = -\frac{1}{4} \partial_\mu \Phi^\alpha \partial^\mu \Phi^\beta. \quad (99)$$

Under a rescaling of the action, $S \mapsto \lambda S$, the charge algebra (81) and the Virasoro algebra with central charge $c = -2$ remain unchanged. However,

$$T(z) \omega(w, \bar{w}) \sim \frac{\lambda}{(z-w)^2} + \frac{\partial \omega(w, \bar{w})}{z-w}, \quad (100)$$

$$\bar{T}(\bar{z}) \omega(w, \bar{w}) \sim \frac{\lambda}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial} \omega(w, \bar{w})}{\bar{z}-\bar{w}}. \quad (101)$$

Perturbing with U thus corresponds to change of relative normalisation of the basis states within each L_0 Jordan block. It only affects the “logarithmic” sector of the Virasoro symmetry and is independent of the $SL(2)$ symmetry. In particular, perturbing an orbifold theory $Z[\Gamma]$ by U will not take us out of the orbifold.

To consider perturbations by the marginal operators V^a let us pick as before a vector \mathbf{n} with $n_1^2 + n_2^2 - n_0^2 = 1$. Perturbing by $V = n_a V^a$ yields the action

$$\begin{aligned} S_\lambda[\Phi^\alpha] &= S + \frac{i\lambda}{2\pi} \int d^2z V \\ &= \int \frac{d^2z}{2\pi} \left\{ \partial\Phi^\alpha \left(J_{\alpha\beta} - \frac{i\lambda}{4} K_{\alpha\beta} \right) \bar{\partial}\Phi^\beta + \bar{\partial}\Phi^\alpha \left(J_{\alpha\beta} + \frac{i\lambda}{4} K_{\alpha\beta} \right) \partial\Phi^\beta \right\}, \end{aligned} \quad (102)$$

where $K_{\alpha\beta} = n_a d_{\alpha\beta}^a$. The equations of motion and the propagator are the same as for the unperturbed theory. In fact, the perturbed action can be related to the original action by a field redefinition. Define new fields

$$\tilde{\Phi}^\alpha(z, \bar{z}) = (e^{i\Lambda K})^\alpha_\beta \phi^\beta(z) + (e^{-i\Lambda K})^\alpha_\beta \bar{\phi}^\beta(\bar{z}), \quad (103)$$

where $\Phi^\beta(z, \bar{z}) = \phi^\beta(z) + \bar{\phi}^\beta(\bar{z})$. We then obtain

$$S[\tilde{\Phi}^\alpha] = \cos \Lambda \int \frac{d^2z}{2\pi} \left\{ \partial\Phi^\alpha (J_{\alpha\beta} + i \tan \Lambda K_{\alpha\beta}) \bar{\partial}\Phi^\beta + \bar{\partial}\Phi^\alpha (J_{\alpha\beta} - i \tan \Lambda K_{\alpha\beta}) \partial\Phi^\beta \right\} \quad (104)$$

Thus, $S_\lambda[\tilde{\Phi}^\alpha]$ is the same as the original action $S[\Phi^\alpha]$ with the identification $\Lambda = -\arctan(\lambda/4)$, apart from a rescaling by $\cos \Lambda = (\lambda^2 + 1/16)^{-1/2}$. Hence, the perturbation does not change the theory and its only effect is a change of basis from Φ to $\tilde{\Phi}$ which can be accomplished by acting with the operator $\exp(i\Lambda Q_-)$, where

$$Q_- = K_{\alpha\beta} \sum_{k=1}^{\infty} \left(\frac{\psi_{-k}^\alpha \psi_k^\beta}{k} - \frac{\bar{\psi}_{-k}^\alpha \bar{\psi}_k^\beta}{k} \right). \quad (105)$$

We expect that perturbing by \tilde{U}, \tilde{V}^a again only results in internal transformations of the theory since the operators \tilde{U}, \tilde{V}^a are the “logarithmic” versions of U, V^a .

The marginal operators in the orbifold models $Z[\Gamma]$ are those marginal operators of the \mathcal{C}_2 model which are invariant under Γ . Since U and \tilde{U} are $SL(2)$ singlets they survive for any Γ . The cyclic orbifolds $Z[\mathcal{C}_{2N}]$, $N > 1$ contain in addition the two operators $V = n_a V^a$ and $\tilde{V} = n_a \tilde{V}^a$. However, these marginal operators effect only internal isomorphisms and thus the orbifold models are isolated theories not linked by any $c = -2$ marginal flows.

There is a simple argument that the \mathcal{C}_{2N} orbifolds are not linked by a marginal flow along the circle line of Gaussian model partition functions even though all the \mathcal{C}_{2N} orbifold partition functions $Z[\mathcal{C}_{2N}](\tau) = Z(n\sqrt{2}, \tau)$ lie on that line. For $Z(\rho, \tau)$ to describe a $c = -2$ model it needs at least a vacuum state with $h = \bar{h} = 0$. This corresponds to a term of the form $|\eta(\tau)^{-1} q^{1/8}|^2$ in the partition function (19). Thus we need $(m\rho + n/\rho)^2 = (m\rho - n/\rho)^2 = 1/2$ for some integers m and n which can only be satisfied if $\rho = n\sqrt{2}$ or $\rho = 1/(m\sqrt{2})$. The circle line of the Gaussian model can, however, be interpreted for each $Z[\mathcal{C}_{2N}]$ model as a flow with the central charge varying with radius as $c = 1 - 6N^2/\rho^2$ but leaving the effective central charge invariant, $c_{\text{eff}} = 1$. For $\rho \rightarrow 1$ we flow to the $SU(2)$ level one WZW model with a state of weight $N^2/4$ taken as the vacuum state.

5 Discussion

We discussed two different conformal field theories at $c = -2$ — the (ξ, η) system and the symplectic fermions — and their orbifold theories. Both theories have the same partition function and identical primary field content and contain reducible but indecomposable representations of the Virasoro algebra. The “small” algebra is a chiral algebra contained in both the (ξ, η) system and the symplectic fermions. The fusion of twist fields yields the “logarithmic” operators extending the “small” algebra to a theory of symplectic fermions. These have an $SL(2)$ symmetry and orbifolds with respect to finite subgroups of $SL(2)$ have modular invariant partition functions reminiscent to the classification of $c = 1$ modular invariant partition functions. The partition function for the symplectic fermion theories sees only the semi-simple part of L_0 since the nilpotent part drops out on taking the trace. For an attempt to define a partition function, based on a different notion of trace, which explicitly displays the logarithmic behaviour see [14]. The orbifold models possess marginal operators which, however, only generate internal symmetries and do not result in flows between different orbifold models.

Fusion in conformal field theory can be formulated algebraically using the notion of a ring-like tensor product of representations of the chiral algebra introduced by Borchers and developed in [23], see also [24]. We intend to give an account of structure and fusion of “logarithmic” representations at $c = -2$ in a forthcoming communication [25]. While “logarithmic” fields can be studied as representations of the Virasoro algebra using the methods presented here their interpretation in an operator formalism and physical significance remain unclear. The rôle of the ground state degeneracy and the correct renormalisation procedure of composite operators need to be investigated further.

The significance of this study goes beyond the $c = -2$ case: For any $c = 1 - 6(p-p')^2/pp'$ from the minimal series, apart from the usual minimal model, there is also a conformal field theory containing “logarithmic” operators. If the Virasoro highest weight representation generated from the vacuum is irreducible the proof of Feigin and Fuchs [26] of the Virasoro fusion rules applies and one obtains a minimal model. But if one includes fields *from the edge* of the Kac-table the vacuum representation becomes reducible and the appearance of “logarithmic” operators is unavoidable. This indicates the possibility of a wide range of two-dimensional critical phenomena where it is not sufficient to restrict to scaling fields alone but where the product of scaling fields yields logarithmic deviations from scaling.

The case $c = -2$ is of interest also in two-dimensional quantum gravity since a $c = -2$ matter system coupled to gravity is exactly solvable [27, 28] in terms of a matrix model [29, 30]. The action for the matter system is precisely the action for the symplectic fermions presented here. An interesting question is whether the logarithmic scaling violations observed in the matrix model [30] are related to the “logarithmic” operators of the symplectic fermions.

Let us finally consider the implications for two dimensional polymers. In [8] H. Saleur argued that the dense phase of two-dimensional polymers could be described by a (ξ, η) -system. Specifically, the (ξ, η) -system describes the sector formed by an even number

of non-contractible polymers with modular invariant partition function $Z_{\text{even}} = Z_1$.² In addition there is a sector formed by an odd number of non-contractible polymers with partition function $Z_{\text{odd}} = Z_2 - Z_1$ which corresponds to a \mathbf{Z}_4 twist of the (ξ, η) -system. The total polymer partition function is the partition function Z_2 of the \mathcal{C}_4 -orbifold of the (ξ, η) -system. This result was obtained by realising dense polymers as the $n \rightarrow 0$ limit of the low temperature phase of the $O(n)$ model which in turn can be mapped onto a Coulomb gas. It agrees with the scaling dimensions

$$x_L^D = \frac{L^2 - 4}{16}. \quad (106)$$

for the polymer L -leg operators Φ_L found by Duplantier [31]. However, the limit $n \rightarrow 0$ does not commute with the thermodynamic limit. Furthermore, the physical quantities which have been determined for dense polymers, the partition function Z_2 and the scaling dimensions (106), are shared by the \mathcal{C}_4 orbifold of both the (ξ, η) system and the symplectic fermions. The Virasoro *primary* field content of both models is the same, they differ only in the reducible Virasoro representations which are unavoidably present. A quantum field theory description of dense polymers will require a new class of fields which form reducible representations of the Virasoro algebra. It is an interesting question whether one can see these fields in a lattice model of polymers and whether one can then distinguish between the twisted (ξ, η) system and symplectic fermions. Numerical studies on the lattice are likely to be difficult since one has to determine 4pt functions in order to see these reducible Virasoro representations. A further interesting question is whether one can see the $SL(2)$ symmetry of the Neveu-Schwarz and Ramond sectors in the geometrical description of the lattice model. These considerations apply equally to the dilute phase of polymers and percolation problems since these are described by a non-minimal $c = 0$ conformal field theory which allows “logarithmic” operators.

Acknowledgements I would like to thank W. Eholzer, M. Flohr, M. Gaberdiel and W. Nahm for many useful discussions. This work has been supported by a Research Fellowship from Sidney Sussex College, Cambridge and partly by PPARC. Explicit calculations were performed on computers purchased on EPSRC grant GR/J73322 using a MAPLE programme written by the author. I also thank the Physikalisches Institut der Universität Bonn for support and hospitality.

A Theta functions

The characters $d_{\mu, \lambda}$ have symmetry properties

$$d_{\mu+1, \lambda}(\tau) = e^{-2\pi i \lambda} d_{\mu, \lambda}(\tau), \quad (107)$$

²There is an error in eq. (37) of [8] which states $Z_{\text{even}} = \frac{1}{2}Z_1$. However, since the partition function Z_1 has a unique ground state it can not be divided by two and it can be easily checked that the expression given here is correct and indeed follows from eq. (24) in [8]

$$d_{\mu,\lambda+1}(\tau) = d_{\mu,\lambda}(\tau), \quad (108)$$

$$d_{-\mu,-\lambda}(\tau) = e^{-2\pi i \mu} d_{\mu,\lambda}(\tau). \quad (109)$$

For N even we can write $d_{k/N,l/N}(\tau)$ as a sum of theta functions,

$$d_{k/N,l/N}(\tau) = \eta(\tau)^{-1} e^{-2\pi i k l / N^2} \sum_{n=0}^{N-1} e^{2\pi i k n / N} \Theta_{Nn+l-N/2, N^2/2}(\tau), \quad (110)$$

while for N odd we have

$$\begin{aligned} d_{k/N+\epsilon/2, l/N+\epsilon'/2}(\tau) &= \eta(\tau)^{-1} e^{-\pi i (2k+N\epsilon)(2l+N\epsilon')/2N^2} \times \\ &\times \sum_{n=0}^{2N-1} e^{2\pi i k n / N} \left(\Theta_{2Nn+2l+(\epsilon'-1)N, 2N^2}(\tau) + \right. \\ &\quad \left. (-1)^\epsilon \Theta_{2N^2+2Nn+2l+(\epsilon'-1)N, 2N^2}(\tau) \right), \end{aligned} \quad (111)$$

where ϵ, ϵ' are zero or one and label the different spin structures. Under $T: \tau \mapsto \tau + 1$ we have

$$d_{\mu,\lambda}(\tau + 1) = e^{2\pi i (\lambda(\lambda-1)/2 + 1/12)} d_{\mu+\lambda, \lambda}(\tau). \quad (112)$$

For the transformation under $S: \tau \mapsto -\tau^{-1}$ we have

$$\eta(-\tau^{-1})^{-1} \Theta_{n,m}(-\tau^{-1}) = \frac{1}{\sqrt{2m}} \sum_{n'=0}^{2m-1} e^{-\pi i n n' / m} \eta(\tau)^{-1} \Theta_{n',m}(\tau) \quad (113)$$

and thus for N even we obtain

$$d_{k/N,l/N}(-\tau^{-1}) = e^{\pi i / 2} e^{-2\pi i (k/N+1/2)(l/N)} d_{1/2-l/N, 1/2+k/N}(\tau). \quad (114)$$

B The Lie algebra $sl(2)$

For the convenience of the reader we list here relations for the Lie algebra $sl(2)$ used above. We use Greek letters $(\alpha, \beta, \gamma, \dots)$ for indices referring to the fundamental representation of $sl(2)$ and letters (a, b, c, \dots) for Lie algebra indices. All relations written in purely tensorial notation are independent of the choice of basis.

The generators of $sl(2)$ are 2×2 matrices $(d^a)_\alpha^\beta$ satisfying

$$[d^a, d^b] = f_c^{ab} d^c, \quad (115)$$

where f_c^{ab} are the structure constants of $sl(2)$ in the chosen basis. The Killing metric on $sl(2)$ is defined as

$$g^{ab} = \text{tr}(d^a d^b) = d_\alpha^{a\beta} d_\beta^{b\alpha}. \quad (116)$$

The symplectic form $J^{\alpha\beta}$ and its inverse $J_{\alpha\beta}$ can be used to raise and lower indices,

$$d_{\alpha\beta}^a = d_\alpha^{a\gamma} J_{\gamma\beta}, \quad d^{a\alpha\beta} = J^{\alpha\gamma} d_\gamma^{a\beta} \quad (117)$$

and analogously for the metric g^{ab} and its inverse g_{ab} . The product of two representation matrices is

$$d_{\alpha}^{a\beta} d_{\beta}^{b\gamma} = d_{\alpha\beta}^a d^{b\beta\gamma} = \frac{1}{2} f_c^{ab} d_{\alpha}^{c\gamma} + \frac{1}{2} g^{ab} \delta_{\alpha}^{\gamma}. \quad (118)$$

Since the square of the fundamental representation decomposes into the trivial and adjoint representation we also have

$$\delta_{\alpha}^{\gamma} \delta_{\beta}^{\delta} = -\frac{1}{2} J_{\alpha\beta} J^{\gamma\delta} + d_{\alpha\beta}^a d_a^{\gamma\delta}. \quad (119)$$

Contracting the above relation yields

$$d_{a\alpha}^{\beta} d_{\beta}^{a\gamma} = \frac{3}{2} \delta_{\alpha}^{\gamma}. \quad (120)$$

A convenient basis to use is the orthonormal basis

$$d_{\alpha}^{0\beta} = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad d_{\alpha}^{1\beta} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad d_{\alpha}^{2\beta} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with commutation relations

$$[d^a, d^b] = \epsilon^{abc} \eta_{cd} d^d, \quad (121)$$

where $\eta_{ab} = \text{diag}(-1, 1, 1)$. The compact generator d^0 is anti-symmetric while the non-compact generators d^1, d^2 are symmetric. The matrices with both indices up or down read in this basis

$$d_{\alpha\beta}^0 = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad d_{\alpha\beta}^1 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad d_{\alpha\beta}^2 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \quad (122)$$

An normalised $sl(2)$ element can be written as

$$K_{\alpha}^{\beta} = n_i d_{\alpha}^{i\beta}, \quad (123)$$

where \mathbf{n} is a unit-vector with respect to the metric η , $n_1^2 + n_2^2 - n_0^2 = 1$. Such an element K satisfies

$$K_{\alpha}^{\beta} K_{\beta}^{\gamma} = \frac{1}{4} \delta_{\alpha}^{\gamma}, \quad \left(e^{i\Lambda K} \right)_{\alpha}^{\beta} = \cos \frac{\Lambda}{2} \delta_{\alpha}^{\beta} + i \sin \frac{\Lambda}{2} K_{\alpha}^{\beta} \quad (124)$$

Furthermore, $K_{\alpha\beta} = K_{\alpha}^{\gamma} J_{\gamma\beta}$ is symmetric. From this we obtain

$$\left(e^{i\Lambda K} \right)_{\alpha}^{\gamma} J_{\gamma\delta} \left(e^{-i\Lambda K} \right)_{\beta}^{\delta} = \cos \Lambda J_{\alpha\beta} + i \sin \Lambda K_{\alpha\beta}. \quad (125)$$

References

- [1] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory, Nucl. Phys. B241 (1984) 333.

- [2] A. Cappelli, C. Itzykson and J.-B. Zuber, The A-D-E Classification of Minimal and $A_1^{(1)}$ Conformal Field Theories, *Commun. Math. Phys.* 113 (1987) 1.
- [3] A. Cappelli, C. Itzykson and J.-B. Zuber, Modular Invariant Partition Functions in Two Dimensions, *Nucl. Phys. B*280 [FS18] (1987) 445.
- [4] R. Dijkgraaf, C. Vafa, E. Verlinde and H. Verlinde, The Operator Algebra of Orbifold Models, *Commun. Math. Phys.* 123 (1989) 485.
- [5] J. McKay, Cartan Matrices, Finite Groups of Quaternions and Kleinian Singularities, *Proceedings of the American Mathematical Society* 81 No. 1 (1981) 153.
- [6] P. Ginsparg, Curiosities at $c = 1$, *Nucl. Phys. B*295 (1988) 153.
- [7] G. Harris, $SU(2)$ Current Algebra Orbifolds of the Gaussian Model, *Nucl. Phys. B*300 (1987) 588.
- [8] H. Saleur, Polymers and percolation in two-dimensions and twisted $N = 2$ supersymmetry, *Nucl. Phys. B*382 (1992) 486, [hep-th/9111007](#).
- [9] D. Friedan, E. Martinec and S. Shenker, Conformal Invariance, Supersymmetry and String Theory, *Nucl. Phys. B*271 (1986) 93.
- [10] V. Gurarie, Logarithmic Operators in Conformal Field Theory, *Nucl. Phys. B*410 [FS] (1993) 535, [hep-th/9303160](#).
- [11] C. Itzykson and J.-M. Drouffe, *Statistical Field Theory*, Cambridge University Press, 1989.
- [12] G. Felder, BRST Approach to Minimal Models, *Nucl. Phys. B*317 (1989) 215, Erratum *ibid.* B324 (1989) 548.
- [13] H. G. Kausch, Extended conformal algebras generated by a multiplet of primary fields, *Phys. Lett.* 259B (1991) 448.
- [14] M. A. I. Flohr, On Modular Invariant Partition Functions of Conformal Field Theories with Logarithmic Operators preprint CSIC-IMAFF-42-1995, [hep-th/9509166](#).
- [15] B. L. Feigin and D. B. Fuchs, Representations of the Virasoro Algebra, *in: Representation of Lie groups and related topics*, eds. A. M. Vershik and D. P. Zhelobenko, pages 465–557, Gordon & Breach, New York, 1990.
- [16] V. S. Dotsenko and V. A. Fateev, Conformal Algebra and Multipoint Correlation Functions in 2D Statistical Models, *Nucl. Phys. B*240 [FS12] (1984) 312.
- [17] L. Dixon, D. Friedan, E. Martinec and S. Shenker, The Conformal Field Theory of Orbifolds, *Nucl. Phys. B*282 (1987) 13.

- [18] L. Dixon, J. Harvey, C. Vafa and E. Witten, Strings on Orbifolds, Nucl. Phys. B261 (1985) 678.
- [19] L. Dixon, J. Harvey, C. Vafa and E. Witten, Strings on Orbifolds. 2., Nucl. Phys. B274 (1986) 285.
- [20] D. S. Freed and C. Vafa, Global Anomalies on Orbifolds, Commun. Math. Phys. 110 (1987) 349, Addendum *ibid.* 117 (1988) 349.
- [21] L. P. Kadanoff, Multicritical behaviour at the Kosterlitz-Thouless critical point, Ann. Phys. 120 (1979) 39.
- [22] L. P. Kadanoff and A. C. Brown, Correlation functions on the critical lines of the Baxter and Ashkin-Teller models, Ann. Phys. 121 (1979) 318.
- [23] M. R. Gaberdiel, Fusion in conformal field theory as the tensor product of the symmetry algebra, Int. J. Mod. Phys. A9 (1994) 4619, [hep-th/9307183](#).
- [24] W. Nahm, Quasi-rational fusion products, Int. J. Mod. Phys. B8 (1994) 3693, [hep-th/9402039](#).
- [25] M. R. Gaberdiel and H. G. Kausch, in preparation.
- [26] B. L. Feigin and D. B. Fuchs, Cohomology of some nilpotent subalgebras of the Virasoro and Kac-Moody Lie algebras, J. Geom. Phys. 5 No. 2 (1988) 209.
- [27] V. A. Kazakov, I. K. Kostov and A. A. Migdal, Critical properties of randomly triangulated planar random surfaces, Phys. Lett. 157B (1985) 295.
- [28] D. V. Boulatov, V. A. Kazakov, I. K. Kostov and A. A. Migdal, Analytical and numerical study of a model of dynamically triangulated random surfaces, Nucl. Phys. B275 (1986) 641.
- [29] F. David, Randomly triangulated surfaces in -2 dimensions, Phys. Lett. 159B (1985) 303.
- [30] I. R. Klebanov and R. B. Wilkinson, Matrix model in -2 dimensions and its effective field theory, Phys. Lett. 251B (1990) 379.
- [31] B. Duplantier, Exact critical exponents for two-dimensional dense polymers, J. Phys. A19 (1986) L1009.